Technical Comments

Comment on "Torsional Oscillation of an Encased Hollow Cylinder of Finite Length"

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ACHENBACH[1] has investigated the torsional motion of a hollow isotropic, elastic cylinder that is encased in a thin elastic shell. It was assumed that the tangential stress and circumferential displacement at the core-casing interface are continuous. Due to the presence of the casing, a time derivative of the displacement $v(r,z,t)$ appears in the boundary conditions (A12)† of Ref. 1, and consequently the associated free vibration problem is of a nonclassical type. After careful study of Ref. 1, the present writer has concluded that Achenbach's representation for the circumferential displacement given in Eqs. (A58) and (A63) satisfies neither the condition of vanishing shear stress on the inner curved surface [Eq. (A13)] nor the boundary condition at the cylinder-casing interface [Eq. (A12)].

In many practical applications of structures of the type under consideration here it may be argued that the stiffness term in the boundary condition in Eq. (A12) $G_s h(\partial^2 v/\partial z^2)$ for the casing dominates the inertia term $\rho_s h(\partial^2 v/\partial t^2)$. Consequently, the effect of the inertia term may be ignored, thereby simplifying the analysis of the forced motion problem. On the other hand, there are problems of engineering significance in which the effect of the inertia term may dominate that of the stiffness term (e.g., see Refs. 2 and 3), thus requiring the retention of the term involving time derivatives in the boundary conditions. In such situations special care must be taken to insure that the representation of the solution of the forced motion problem satisfies the boundary conditions.

In the following paragraphs, a representation of the solution of Achenbach's problem that satisfies the equation of motion, the initial conditions, and all the boundary conditions is derived. The method of solution is based upon one of three techniques described by Anderson and Thomas[4] that may be used for solving boundary-initial value problems that possess both prescribed time-dependent boundary conditions and time derivatives of the dependent variable in the boundary conditions. The notation of Ref. 1 will be followed as closely as possible.

To solve the differential equation (A7) subject to the boundary conditions (A12), (A13), (A41), and (A42), and to the initial conditions (A43), we introduce the new dependent variable $w(r,z,t)$ according to (A58). We obtain a new boundary-initial value problem consisting of

$$L[w] + \partial^2 w/\partial z^2 - (1/c_p^2)(\partial^2 w/\partial t^2) = F(r,z,t) \quad (1)$$

$$w(r,0,t) = w(r,l,t) = 0 \quad (2)$$

$$a(\partial w/\partial r) - w = -(z/l)[af' - f]g(t) \text{ on } r = a \quad (3)$$

$$b(\partial w/\partial r) - w - bh(G_s/G_p)(\partial^2 w/\partial z^2) + bh(\rho_s/G_p)(\partial^2 w/\partial t^2) = -(z/l)\{[bf' - f]g(t) - bh(\rho_s/G_p)f\ddot{g}(t)\} \text{ on } r = b \quad (4)$$

$$w(r,z,0) = -(z/l)f(r)g(0) \quad (5)$$

$$\dot{w}(r,z,0) = -(z/l)f(r)\dot{g}(0) \quad (6)$$

where L and $F(r,z,t)$ are given in (A48) and (A60), respectively. We observe that the transformation (A58) leads us to a boundary value problem that has homogeneous conditions, Eq. (2), on the surfaces $z = 0,l$. However, the condition in Eq. (3) for the inner surface of the core and the interface condition in Eq. (4) are no longer homogeneous.

In view of the homogeneous boundary conditions in Eq. (2), we remove the z-dependence from Eq. (1) by applying the finite Fourier sine transform

$$W(r;m;t) = \int_0^l w(r,z,t) \sin\alpha_m z dz \quad (7)$$

with $\alpha_m = m\pi l, m = 1,2,3, \ldots$. The associated inversion formula is

$$w(r,z,t) = \left(\frac{2}{l}\right) \sum_{m=1}^{\infty} W(r;m;t) \sin\alpha_m z dz \quad (8)$$

Applying Eq. (7) in the familiar manner, we can reduce the boundary-value problem in Eqs. (1–6) to the following:

$$L[W] - \alpha_m^2 W - (1/C_p^2)(\partial^2 W/\partial t^2) = \bar{F}(r;m;t) \quad (9)$$

$$a(\partial W/\partial r) - W = S_m(a,t) \text{ on } r = a \quad (10)$$

$$b(\partial W/\partial r) - [1 - bh\alpha_m^2(G_s/G_p)]W + bh(\rho_s/G_p)(\partial^2 W/\partial t^2) = U_m(b,t) \text{ on } r = b \quad (11)$$

$$W(r;m;0) = (-1)^m f(r)g(0)/\alpha_m$$
$$\dot{W}(r;m;0) = (-1)^m f(r)\dot{g}(0)/\alpha_m \quad (12)$$

where

$$\bar{F}(r;m;t) = (-1)^m(1/\alpha_m)\{g(t)L[f] - (1/C_p^2)f(r)\ddot{g}(t)\}$$
$$S_m(r,t) = (-1)^m(1/\alpha_m)[rf' - f]g(t)$$
$$U_m(r,t) = (-1)^m(1/\alpha_m)\{[rf' - f]g(t) - rh(\rho_s/G_p)f\ddot{g}(t)\}$$

It is now necessary to solve Eq. (9) subject to Eqs. (10–12). Because of the presence of the term $\partial^2 W/\partial t^2$ in Eq. (11), extra care must be taken in order to assure that the representation for W indeed does satisfy this boundary condition. A related problem was discussed in Ref. 4, and we shall modify the method used there so as to apply to the present problem. The essential idea is to use a decomposition for W of the Williams' type[5]:

$$W(r;m;t) = V_m(r,t) + \sum_{n=1}^{\infty} R_{nm}(r)T_{nm}(t) \quad (13)$$

where the "quasi-static" part of the solution, V_m, and the generalized coordinates, T_{nm} will be determined later, and R_{nm} are the eigenfunctions given in (A45), which satisfy the homogeneous boundary conditions

$$aR'_{nm}(a) - R_{nm}(a) = 0$$
$$bR'_{nm}(b) - [1 - bh\alpha_m^2(G_s/G_p) + bh(\rho_s/G_p)\omega_{nm}^2]R_{nm}(b) = 0 \quad (14)$$

Note that the eigenvalue ω_{nm}^2 appears in the second boundary condition in Eq. (14). The solution of Achenbach's boundary value problem will be complete formally once V_m and T_{nm} have been determined.

Substitution of Eq. (13) into Eq. (9) leads to

$$L[V_m] - \alpha_m^2 V_m + \sum_{n=1}^{\infty} \{(L[R_{nm}] - \alpha_m^2 R_{nm})T_{nm}(t) - (1/C_p^2)R_{nm}\ddot{T}_{nm}(t)\} = \bar{F}(r;m;t) + (1_1/C_p^2)\ddot{V}_m \quad (15)$$

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† Equation numbers preceded by "A" refer to Ref. 1.

Let

$$L[V_m] - \alpha_m^2 V_m = 0 \quad (16)$$

Then, in view of Eqs. (16) and (A16), Eq. (15) assumes the form

$$\sum_{n=1}^{\infty} [\ddot{T}_{nm}(t) + \omega_{nm}^2 T_{nm}(t)] R_{nm}(r) = -\ddot{V}_m - C_p^2 \bar{F} \quad (17)$$

If we substitute Eq. (13) into the boundary conditions (10) and (11), we find, using Eq. (14),

$$r(\partial V_m/\partial r) - V_m = S_m(r,t), \text{ at } r = a \quad (18)$$

$$r\left(\frac{\partial V_m}{\partial r}\right) - \left[1 - rh\alpha_m^2\left(\frac{G_s}{G_p}\right)\right] V_m + \sum_{n=1}^{\infty} rh\left(\frac{\rho_s}{G_p}\right) R_{nm}\ddot{T}_{nm} + \sum_{n=1}^{\infty}\left\{rR'_{nm} - \left[1 - rh\alpha_m^2\left(\frac{G_s}{G_p}\right)\right] R_{nm}\right\} T_{nm} = -rh\left(\frac{\rho_s}{G_p}\right) V_m + U_m \quad (19)$$

at $r = b$. To eliminate $\ddot{T}_{nm}$ from Eq. (19), we set $r = b$ in Eq. (17) and write

$$\sum_{n=1}^{\infty} R_{nm}\ddot{T}_{nm} = -\sum_{n=1}^{\infty} \omega_{nm}^2 R_{nm} T_{nm} - \ddot{V}_m - C_p^2 \bar{F} \quad (20)$$

Upon substitution of Eq. (20) into Eq. (19), we obtain, by virtue of the second condition in Eq. (14)

$$r(\partial V_m/\partial r) - [1 - rh\alpha_m^2(G_s/G_p)]V_m = U_m^*(r,t) \quad \text{at } r = b \quad (21)$$

where

$$U_m^*(r,t) = rh(\rho_s/\rho_p)\bar{F}(b;m;t) + U_m(r,t)$$

The solution of Eq. (16) is

$$V_m(r,t) = A_m^{(1)}(t) I_1(\alpha_m r) + A_m^{(2)}(t) K_1(\alpha_m r) \quad (22)$$

where I_1 and K_1 are modified Bessel functions of order unity of the first and second kinds, respectively. From the boundary conditions (18) and (21), we find $A_m^{(j)}(t) = C_m^{(j)}(t)/\delta_m, j = 1,2$, with

$$C_m^{(1)}(t) = bS_m(a,t)[K_2(\alpha_m b) - h\alpha_m(G_s/G_p)K_1(\alpha_m b)] - aU_m^*(b,t)K_2(\alpha_m a)$$

$$C_m^{(2)}(t) = aU_m^*(b,t)I_2(\alpha_m a) - bS_m(a,t)[I_2(\alpha_m b) + h\alpha_m(G_s/G_p)I_1(\alpha_m b)]$$

$$\delta_m = ab\alpha_m\{I_2(\alpha_m a)[K_2(\alpha_m b) - h\alpha_m(G_s/G_p)K_1(\alpha_m b)] - K_2(\alpha_m a)[I_2(\alpha_m b) + h\alpha_m(G_s/G_p)I_1(\alpha_m b)]\}$$

Finally, the initial conditions in Eq. (12) may be written as

$$\sum_{n=1}^{\infty} R_{nm}(r)T_{nm}(0) = -V_m(r,0) + \frac{(-1)^m f(r)g(0)}{\alpha_m}.$$

$$\sum_{n=1}^{\infty} R_{nm}(r)\dot{T}_{nm}(0) = -\dot{V}_m(r,0) + \frac{(-1)^m f(r)\dot{g}(0)}{\alpha_m} \quad (23)$$

If we apply the orthogonality condition (A54) to Eqs. (17) and (23), we obtain the following initial value problem:

$$\ddot{T}_{nm}(t) + \omega_{nm}^2 T_{nm}(t) = \mathcal{R}_{nm}(t)$$

$$T_{nm}(0) = -\bar{V}_m(n;0) + (-1)^m \bar{f}(n)g(0)/\alpha_m, \quad (24)$$

$$\dot{T}_{nm}(0) = -\dot{\bar{V}}(n;0) + (-1)^m \bar{f}(n)\dot{g}(0)/\alpha_m$$

where

$$\mathcal{R}_{nm}(t) = (R_{nm}, -\ddot{V}_m - C_p^2\bar{F})/(R_{nm},R_{nm})$$

$$\bar{V}_m(n;t) = (R_{nm},V_m)/(R_{nm},R_{nm}), \quad \bar{f}(n) = (R_{nm},f)/(R_{nm},R_{nm})$$

using the notation of Ref. 1 for the right sides of these last two equations.

The solution of the initial value problem in Eq. (24) is

$$T_{nm}(t) = T_{nm}(0)\cos\omega_{nm}t + \left[\frac{\dot{T}_{nm}(0)}{\omega_{nm}}\right]\sin\omega_{nm}t + \left(\frac{1}{\omega_{nm}}\right)\int_0^t \mathcal{R}_{nm}(\tau)\sin\omega_{nm}(t-\tau)d\tau \quad (25)$$

From Eqs. (A58, 8, and 13), it now follows that the circumferential displacement $v(r,z,t)$ is given by

$$v(r,z,t) = \left(\frac{z}{l}\right)f(r)g(t) + \left(\frac{2}{l}\right)\sum_{n=1}^{\infty} V_m(r,t)\sin\alpha_m z + \left(\frac{2}{l}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{nm}(r)T_{nm}(t)\sin\alpha_m z \quad (26)$$

where $f(r)$ and $g(t)$ are prescribed functions, $V_m(r,t)$ is given in Eq. (22), the radial eigenfunctions $R_{nm}(r)$ appear in Eq. (A45), and the generalized coordinates $T_{nm}(t)$ are obtained from Eqs. (25).

Equation (26), then, is a formal representation of the solution of the boundary value problem posed in Ref. 1; it satisfies the equation of motion, all boundary conditions, and all initial conditions.

References

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Reply by Author to G. L. Anderson

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ANDERSON is correct in pointing out that Eqs. (A58) and (A63) strictly satisfy neither the condition of vanishing shear stress on the inner curved surface, Eq. (A13), nor the boundary condition at the cylinder-casing interface, Eq. (A12). He is also correct when he writes that in many practical applications of the structure considered in Ref. 1, it may be argued that the stiffness term $G_s h(\partial^2 v/\partial z^2)$ in the boundary condition Eq.(A12) dominates the inertia term $\rho_s h(\partial^2 v/\partial t^2)$. Thus, for a structure such as an encased solid propellant grain, which was discussed in Ref. 1, the effect of the inertia term may generally be ignored.

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